Extrema Everywhere

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ABSTRACT: What kinds of real-valued functions f have a local extremum at every point?

Entrance. Henceforth let J denote $[0, 1]$.

1: Proposition A (jk). *Suppose $f: J \to \mathbb{R}$ has a local-max or local-min at every point of J. If f is continuous then f is constant.* ♢

Chris Stark, in a more general setting, used my proof of the above to show the theorem below, which he jokingly called "A general topologist's version of Sard's theorem". It implies that the set of extreme-values is countable. So the theorem implies the proposition —since a countably valued continuous function $J \to \mathbb{R}$ must be constant.

A space is ***Lindelöf*** if every open cover has a countable subcover.

2: Theorem B (Stark). *Let X be a Lindelöf metric space and $f: X \to \mathbb{R}$ be a (not necessarily continuous) function. Then the set of local extremal values,*

$$\Big\{ f(x) \;\Big|\; x \text{ is a local-min or local-max of } f \Big\},$$

is a countable subset of the reals. ♢

Proof. It is enough to prove that the set $\mathcal{V} \subset \mathbb{R}$ of local minimal-values is countable. For each $v \in \mathcal{V}$, pick a point $x_v \in X$ which is a local-min of f with $f(x_v) = v$. Each x_v is in the center of some open ball B_v such that $f\downharpoonright_{B_v}$ attains a minimum at x_v.

Assume, FTSOContradiction, that $\mathcal{V}$ is uncountable. Consequently, we can find a radius $\rho > 0$ and drop to an uncountable subset of $\mathcal{V}$ so that, now,

3: For each $v \in \mathcal{V}$: $\text{Radius}(B_v) \geqslant 3\rho$.

We leave it as an exercise to show that each uncountable set E in a Lindelöf metric space satisfies

4: *For any positive ρ there are distinct points $x_1, x_2 \in E$ at distance less than ρ.*

Applying this to $E := \{x_v \mid v \in \mathcal{V}\}$ gives distinct points $x_v, x_w \in E$ such that, by (3), each is in the "basin of local-minimality" of the other. Thus $f(x_v) = f(x_w)$, i.e, $v = w$. This implies the contradiction that $x_v = x_w$. ♦

The proof yields a cardinality assertion about the set of local extremal-*points*, rather than extremal-values.

5: Corollary. *With X and f as above, the subset of X of strict local-extrema is countable.* ♢

N.B. See source file for an extension.

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